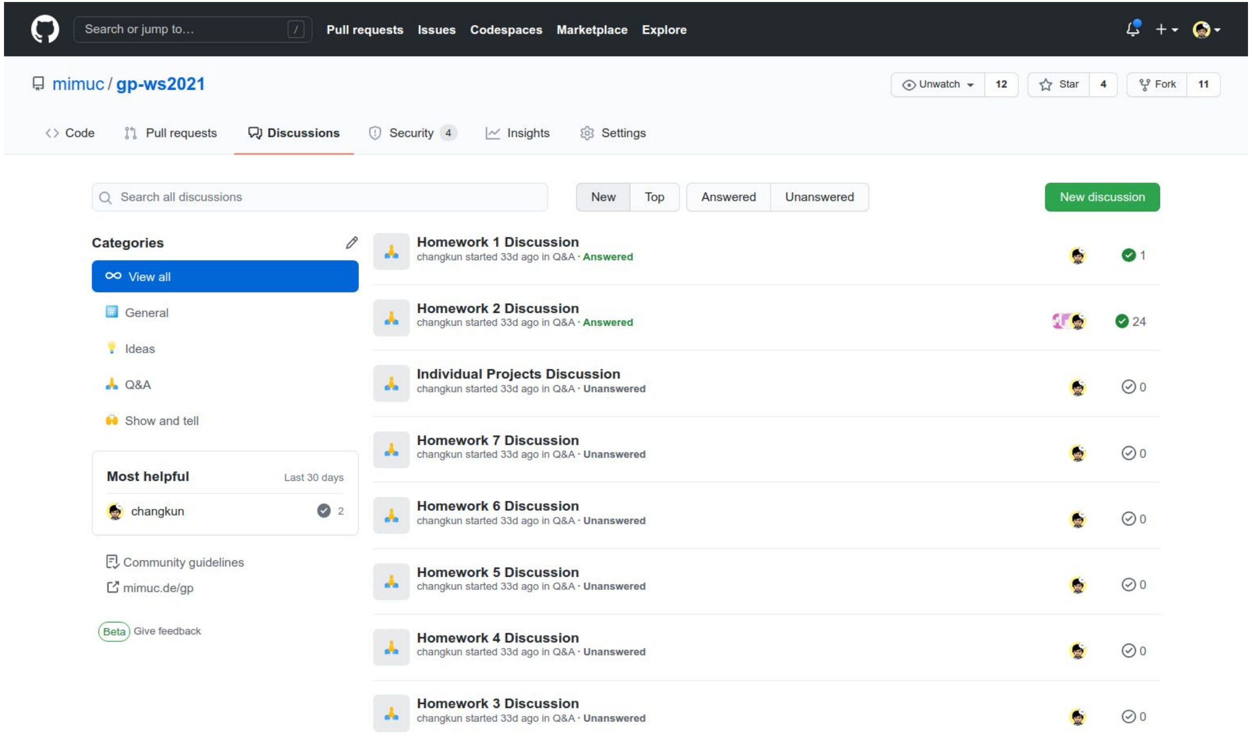
Geometry Processing

4 Parameterization

Ludwig-Maximilians-Universität München

Announcements

- Switch to GitHub Discussions: https://github.com/mimuc/gp/discussions



Session 4: Parameterization

- Motivation
- Methods
  - Tutte's Embedding Theorem (Barycentric Mapping)
  - Least Squares Conformal Maps (LSCM)
  - Angle-based Flattening (ABF)
- Summary
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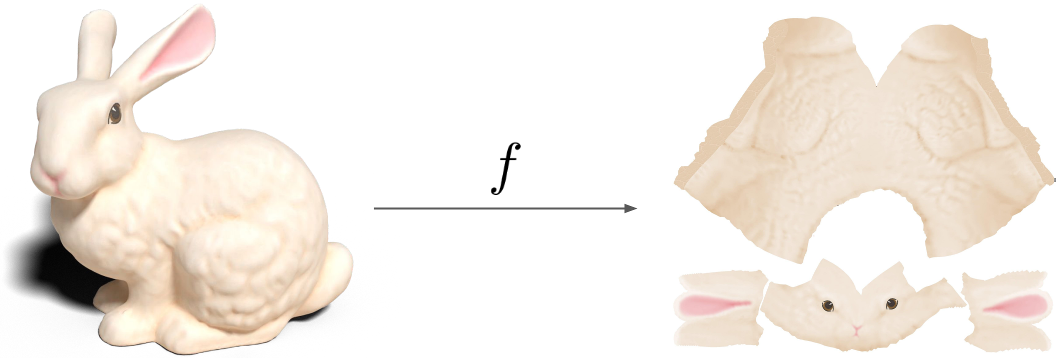
Parameterization: Definition

A function f that maps input surface in ***one-to-one*** correspondence with a different (e.g. 2D) domain



Example: In UV Mapping:

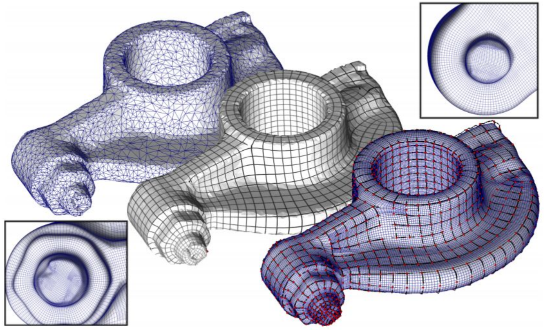
- Each vertex is associated with an UV coordinate $(x_i, y_i, z_i) \rightarrow (u_i, v_i)$
- Caution: vertices at seam

Equivalent terminologies: Flattening, unfolding

Parameterization: Applications

Different types:

- Surface to plane mapping
  - Producing UV/normal/displacement/... maps
  - Compression [Gu et al 2002] (UE5's Virtual Geometry)
  - ...
- Plane to surface mapping
  - Remeshing (later)
  - ...
- Surface to surface mapping
  - Deformation (later)



Session 4: Parameterization

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- **Methods**
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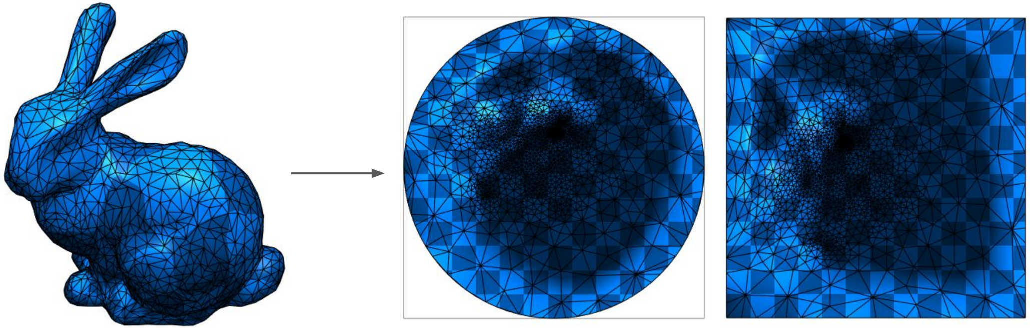
Barycentric Mapping (Tutte's Embedding Theorem [Tutte 1960])

From graph theory: *Given a triangulated surface homeomorphic to a disk, if the (u, v) coordinates at the boundary vertices lie on a convex polygon in order, and if **the coordinates of the internal vertices are a convex combination of their neighbors**, then the (u, v) coordinates from a valid parameterization.*

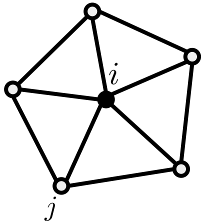


A convex polygon: circle, square, ...

Interior vertices: $\{1, \ldots, n_{\text{int}}\}$ Boundary vertices: $\{n_{\text{int}}+1, \ldots, n\}$

A convex combination (*barycentric coordinates!*):

$$-w_{ii}\begin{pmatrix}u_i\\v_i\end{pmatrix} = \sum_{j\neq i} w_{ij}\begin{pmatrix}u_i\\v_j\end{pmatrix}$$



Barycentric Mapping: Matrix Form

For interior vertices:

$$-w_{ii}\begin{pmatrix}u_i\\v_i\end{pmatrix} = \sum_{j\neq i} w_{ij}\begin{pmatrix}u_i\\v_j\end{pmatrix} \Rightarrow \quad w_{ii} = -\sum_{i\neq j} w_{ij}$$

Recall Laplace matrix:

$$\mathbf{L} = \mathbf{DW}$$
$$\mathbf{W} = (W_{ij}) \qquad W_{ij} = \begin{cases} -\sum_{ik} w_{ik}, & \text{if } i = j \\ w_{ij}, & \text{if } j \text{ is a neighbor of } i \\ 0, & \text{otherwise} \end{cases}$$

All we need is to solve two linear equations: $\mathbf{Lu}' = \mathbf{u} \quad \mathbf{Lv}' = \mathbf{v}$

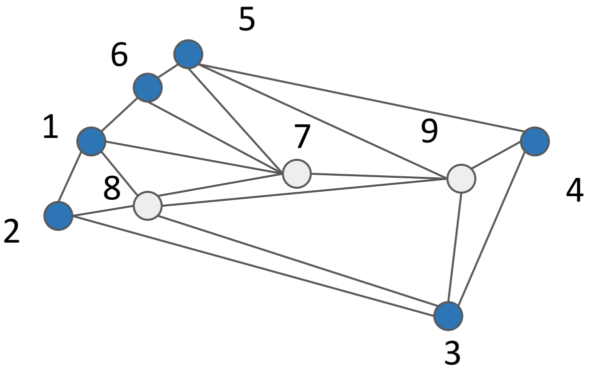
where the elements of $\mathbf{u}$ (respectively $\mathbf{v}$) is either zero (interior vertices) or precomputed (boundary vertices)

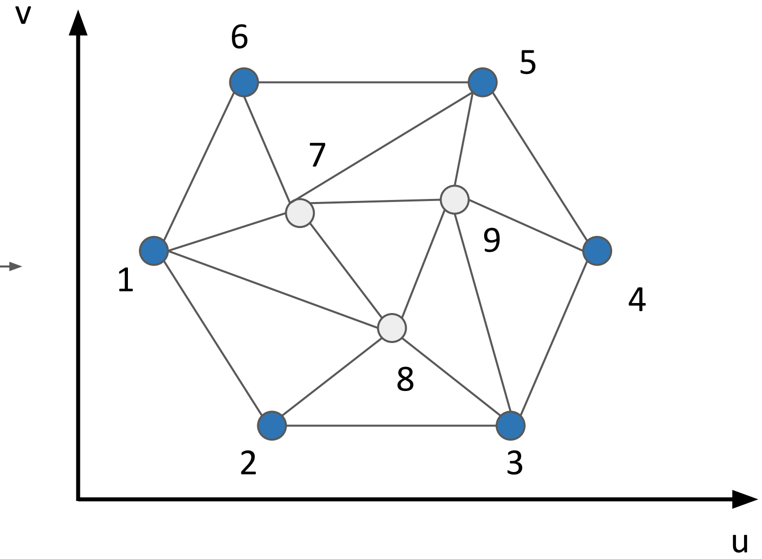
The solution $(\mathbf{u}', \mathbf{v}')$ is the barycentric mapped UV coordinates.

$$\Delta f = 0$$

The Laplace Equation

Barycentric Mapping: Uniform Laplacian as Example





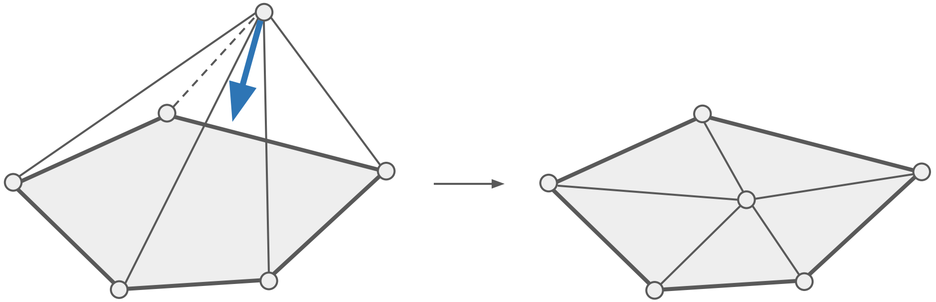
$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & -5 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -5 \end{pmatrix} \quad \mathbf{u}' = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} u_{v1} \\ u_{v2} \\ u_{v3} \\ u_{v4} \\ u_{v5} \\ u_{v6} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{L}\mathbf{u}' = \mathbf{u}$$
$$\mathbf{L}\mathbf{v}' = \mathbf{v}$$

(See homework 4 :)

Barycentric Mapping: Intuition & Issues

Intuitively:

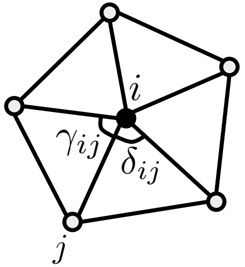


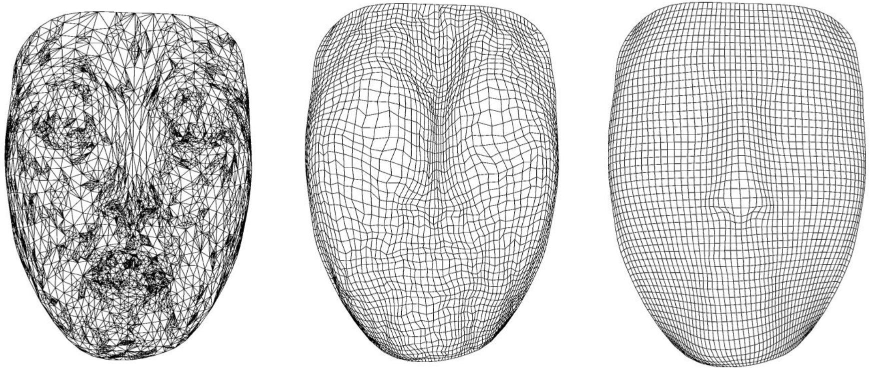
Different choice of Laplace matrix: Uniform, Cotan, ...

Caution: Tutte's embedding theorem requires the Laplacian to satisfy: LOC+LIN+POS (why?)

Cotan Laplacian can violates POS, there is a better version "*mean value weights*" **[Floater 03]** produce provably positive ones:

$$w_{ij} = \frac{1}{||f_i - f_j||}\left(\tan\left(\frac{\gamma_{ij}}{2}\right) + \tan\left(\frac{\delta_{ij}}{2}\right)\right)$$

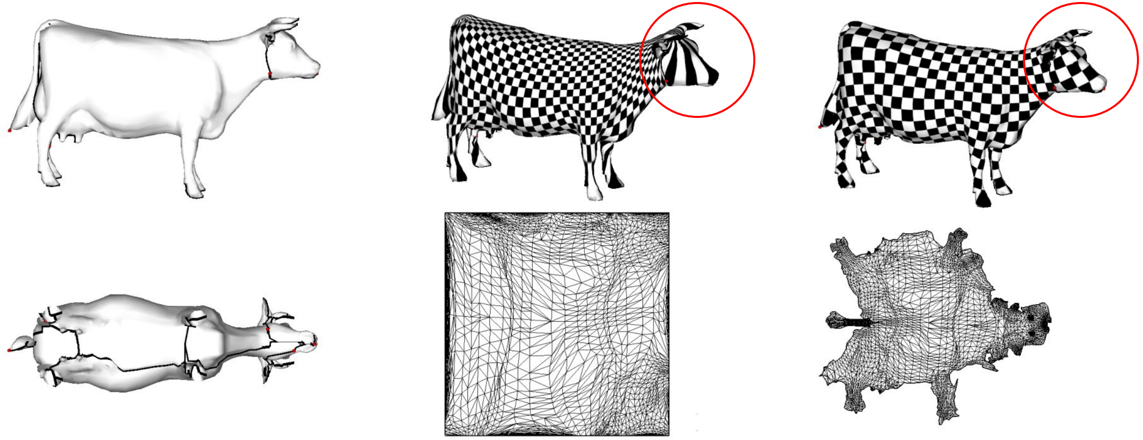




Original Mesh | Uniform Tutte | Mean Value

Barycentric Mapping: Issues (cont.)

Tutte's Embedding requires a fixed convex boundary ⇒ High distortion and mesh must have at least a boundary



Fixed Boundary

Free Boundary

How to minimize the distortion?

How to cut a "watertight" mesh?

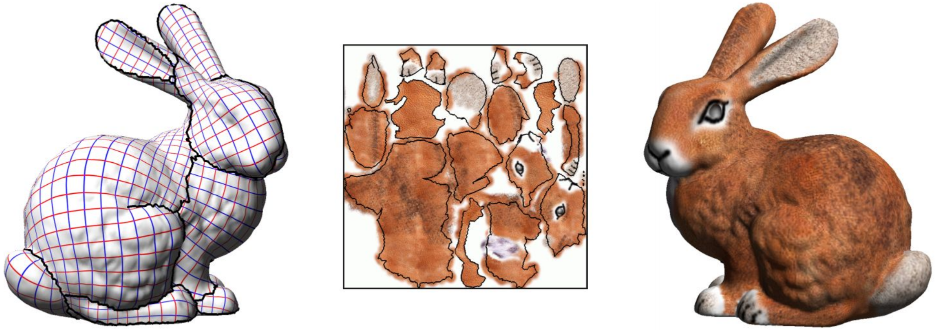
How to achieve free boundary?

Texture Atlas Generation [Lévy et al 2002]

The generation of a texture atlas can be decomposed into the following steps:

1. **Segmentation**: The model is partitioned into a set of charts
2. **Parameterization**: Each chart is 'unfolded', i.e. put in correspondence with a subset of R^2
3. **Packing**: The charts are gathered in texture space.

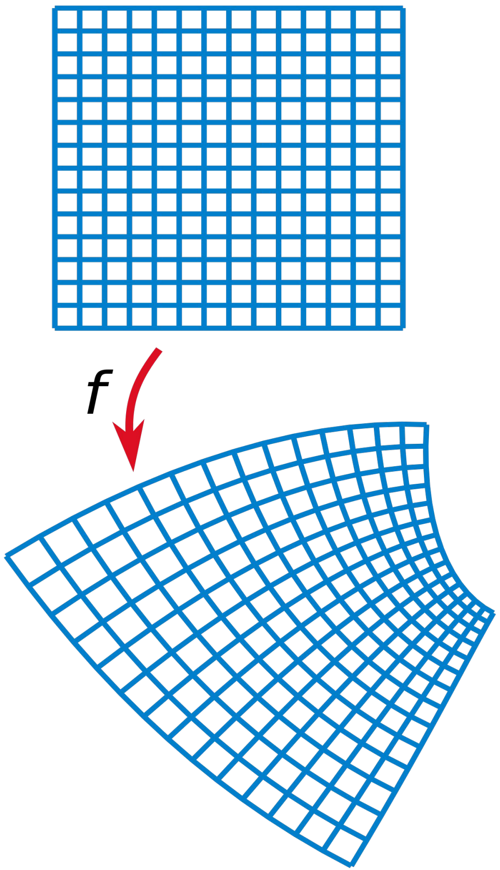
This workflow still largely exists in today's modeling practice (either manual mark seam or "smart" unwrap)



Conformal Mapping

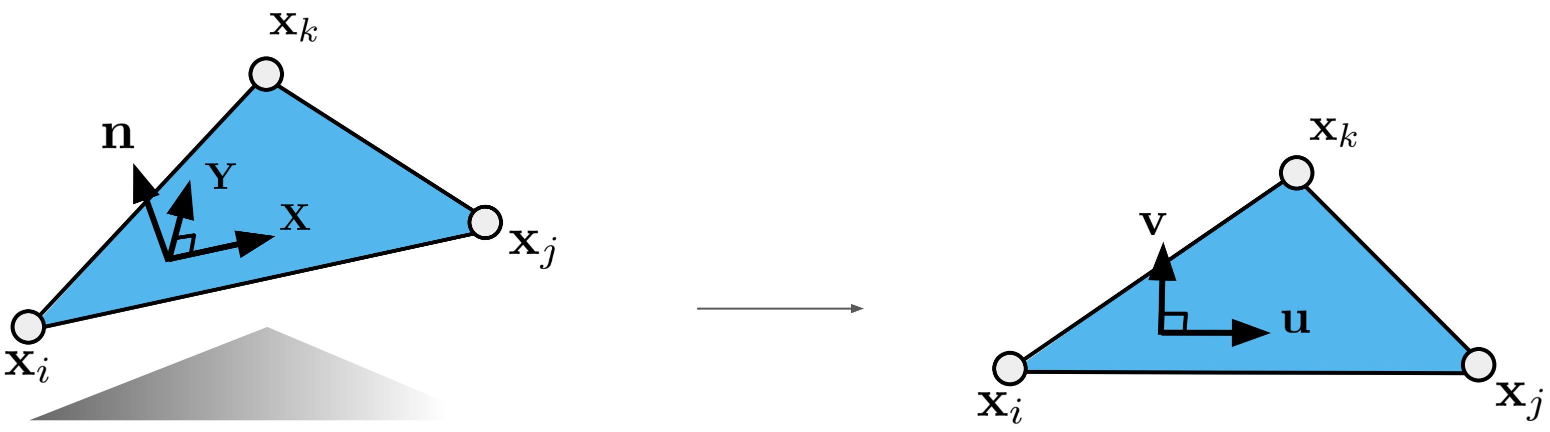
Conformal mappings = rotation + scale

Scale distortion is smoothly distributed (harmonic)



Equivalent terminologies: Angle-preserving, similar, scale

Mapping in General



$$f(u+\Delta u, f+\Delta v) = f(u,v) + f_u(u,v)\Delta u + f_v(u,v)\Delta v \quad \Rightarrow f(u+\Delta u, v+\Delta v) = f(u,v) + \mathbf{J}\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix}$$

To achieve angle preservation, the Jacobian must be a similarity transformation:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = s\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{matrix}$$

(Surprisingly leads to *Cauchy-Riemann Equations*)

Least Squares Conformal Maps (LSCM, ASAP) [Lévy et al 2002]

If the parameterization subject to $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, then it is a conformal, but in generally it is impossible (why?)

Instead, we minimize the least square "energy" as our objective:

$$E_{\mathrm{LSCM}} = \sum_t A_t \left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right)$$

(triangle area)

The energy measures non-conformality

It is invariant with respect to arbitrary translations and rotations

Least Squares Conformal Maps (LSCM, ASAP) [Lévy et al 2002]

Issues:

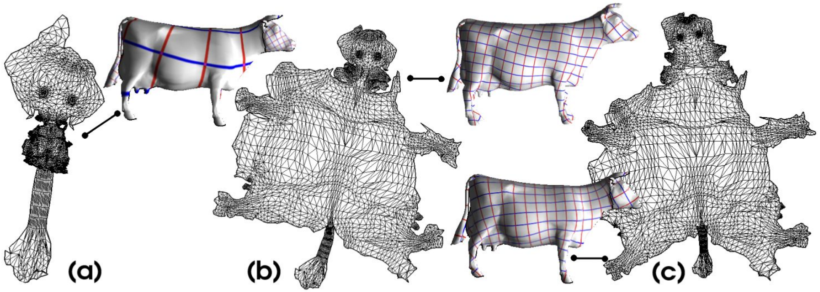
Energy does not have a unique minimizer, one can fix at least two vertices.

The choice of the vertices affects the results significantly

No guarantee on bijective

No guarantee on flip-free

...



Angle-based Flattening (ABF)

More straightforward: Given angles for original mesh, find the closest angle that describe a flat mesh

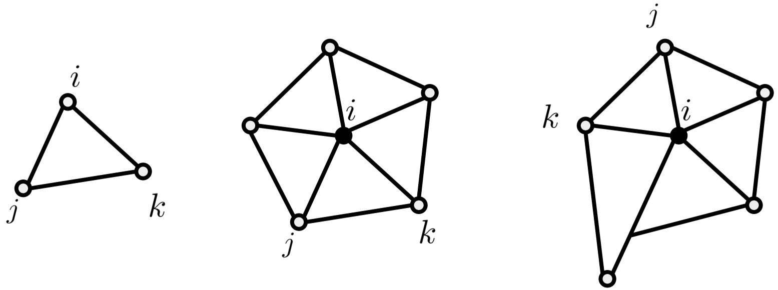
$$\min_{\theta} \sum_{i} (\hat{\theta}_i - \theta_i)^2$$

Subject to

1. angle sum: $\theta_i + \theta_j + \theta_k = \pi$

2. interior vertices sum: $\sum_{ijk} \theta_i = 2\pi$

3. compatible lengths around vertices (law of sines): $\prod_{ijk} \frac{\sin \theta_j}{\sin \theta_k} = 1$



Nonlinear optimization, many approximations: LinearABF [Zayer et al 2007], ABF++ [Sheffer et al 2005], etc.

You have the ability to known how they work exactly and be able to implement them eventually.

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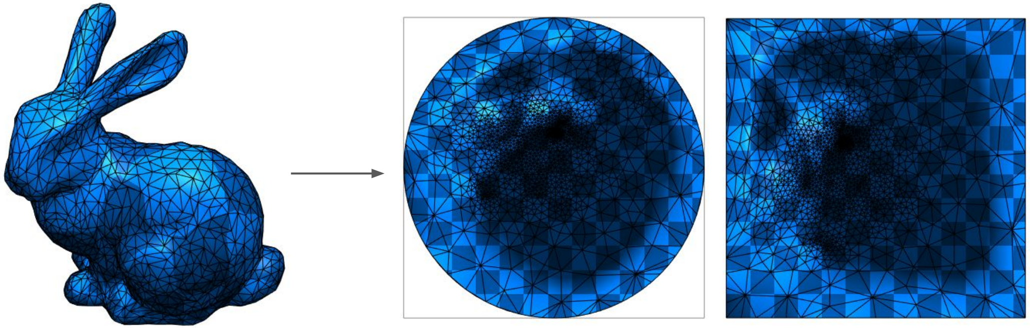
Summary

- Texture atlas generation workflow containing three parts: Segmentation, parameterization, and packing
- Tutte's embedding is a classic baseline for parameterization
- Objectives: **Minimize distortion (any kind of properties) with respect to a certain measure**
  - Validity (bijective, no self intersection): flipped triangle?
  - Boundary: fixed or free?
  - Domain: 2D or 3D?
  - ...
- A very large number of techniques exists that solves different problems (no best solution!)

Homework 4: Tutte's Barycentric Embedding

Implement a barycentric mapping that maps a given bunny mesh to different types of boundary:

1. Compute boundary coordinates in order
2. Construct linear equation for interior and boundary vertices for uniform and cotan Laplacian (optional: mean value)
3. Solve the linear equation and update UV coordinates of each vertices



More details: https://github.com/mimuc/gp/blob/ws2021/homeworks/4-param

Discussion panel: https://github.com/mimuc/gp/discussions/4

Submission Instructions: https://github.com/mimuc/gp/tree/ws2021/homeworks#submission-instruction

Further Readings

[Tutte 1960] (Tutte Embedding) Tutte WT. Convex representations of graphs. Proceedings of the London Mathematical Society. 1960.

[Lévy et al 2002] (LSCM) Lévy B, Petitjean S, Ray N, Maillot J. Least squares conformal maps for automatic texture atlas generation. ACM transactions on graphics (TOG). 2002.

[Sheffer et al 2001] (ABF) Sheffer A, de Sturler E. Parameterization of faceted surfaces for meshing using angle-based flattening. Engineering with computers. 2001.

[Gu et al 2002] Gu X, Gortler SJ, Hoppe H. Geometry images. InProceedings of the 29th annual conference on Computer graphics and interactive techniques 2002 Jul 1.

[Floater 03] Floater MS. Mean value coordinates. Computer aided geometric design. 2003 Mar.

[Sheffer et al 2005] (ABF++) Sheffer A, Lévy B, Mogilnitsky M, Bogomyakov A. ABF++: fast and robust angle based flattening. ACM Transactions on Graphics (TOG). 2005.

[Zayer et al 2007] Zayer R, Lévy B, Seidel HP. Linear angle based parameterization. in Eurographics SGP, Jul 2007.

[Hormann et al 2007] Kai Hormann, Bruno Lévy, and Alla Sheffer. Mesh parameterization: theory and practice Video files associated with this course are available from the citation page. In ACM SIGGRAPH Courses. 2007.

[Smith et al 2015] Smith J, Schaefer S. Bijective parameterization with free boundaries. ACM Transactions on Graphics (TOG). 2015 Jul 27.

Thanks! What are your questions?

Next session: Remeshing

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- Discussion: Parameterization in Blender & Homework 3 Implementation

Mesh Parameterization in Blender

User Manual

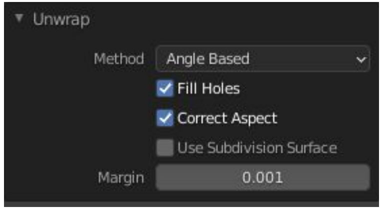
- https://docs.blender.org/manual/en/latest/modeling/meshes/uv/unwrapping/introduction.html
- https://docs.blender.org/manual/en/latest/modeling/meshes/editing/uv.html#unwrap
- https://docs.blender.org/manual/en/latest/modeling/meshes/editing/uv.html#smart-uv-project

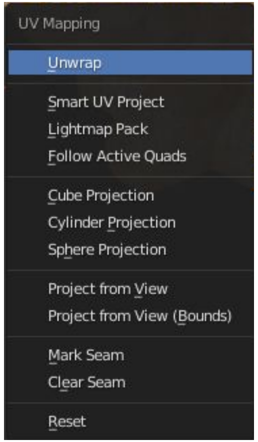
API Manual

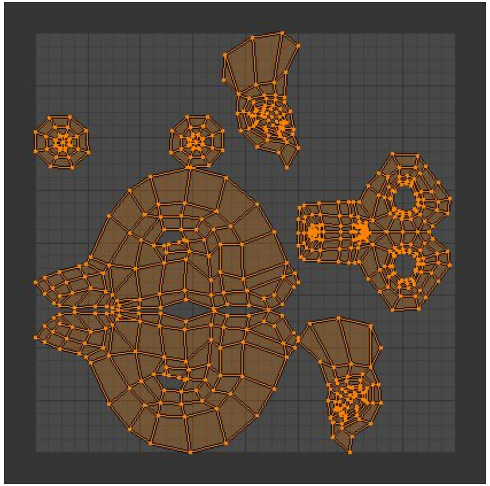
- https://docs.blender.org/api/current/bpy.ops.uv.html?highlight=uv#bpy.ops.uv.unwrap

Implementation

- source/blender/editors/uvedit/uvedit_parameterizer.c (c2a01a6c118e)







Homework 3: Laplace Matrix

```
laplaceWeightMatrix(weightType) {
   const n = this.vertices.length
   let T = new Triplet(n, n)
   for (const vert of this.vertices) {
     const i = vert.idx
     let sum = 1e-8 // Tikhonov regularization to get strict positive definite
     vert.halfedges(h => {
       let w = 0
       switch (weightType) {
       case 'uniform':
         w = 1
         break
       case 'cotan':
         w = (h.cotan() + h.twin.cotan())/2
       }
       sum += w
       T.addEntry(-w, i, h.twin.vertex.idx)
     })
     T.addEntry(sum, i, i)
   }
   return SparseMatrix.fromTriplet(T)
 }
```

Blender: Geometry Nodes

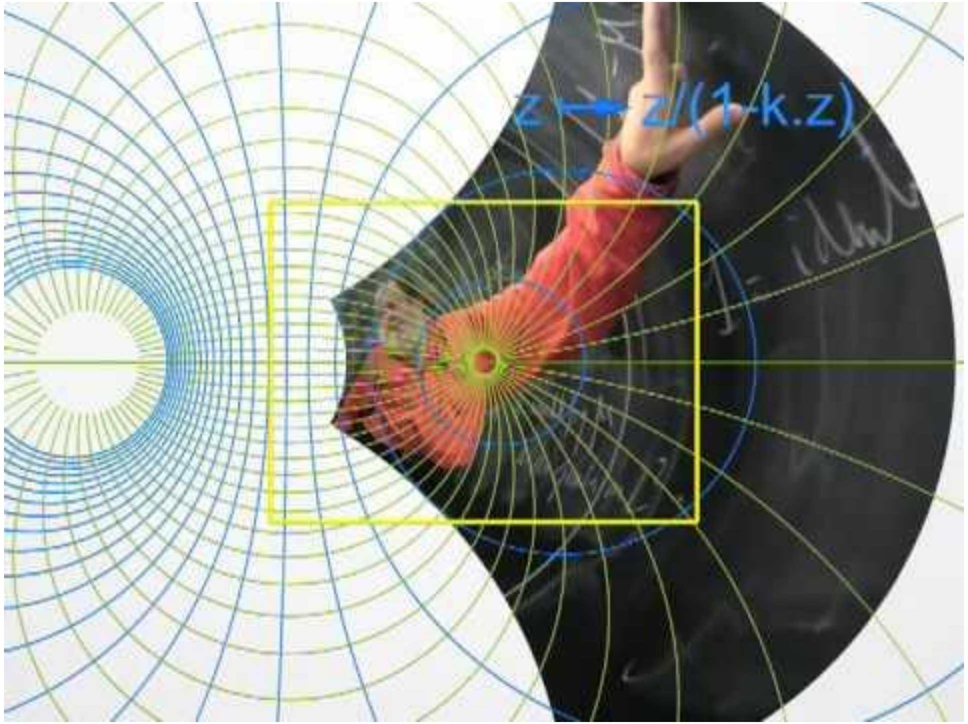
Beta release: https://builder.blender.org/download/

Manual: https://docs.blender.org/manual/en/dev/modeling/modifiers/nodes/index.html

Workboard: https://developer.blender.org/project/board/121/

Dimensions: A walk through mathematics (2011)

https://www.youtube.com/watch?v=yJZP_-40KVw&list=PL97CCC2CC4E89C7E5



https://www.youtube.com/watch?v=MWHMzgZ4VIk

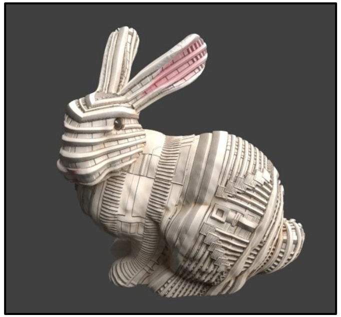
Backlog

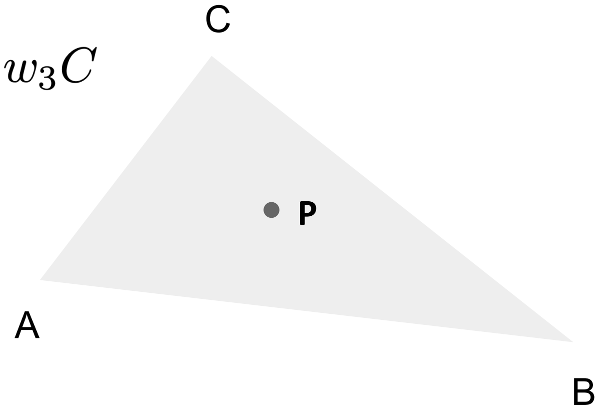
random mind trash

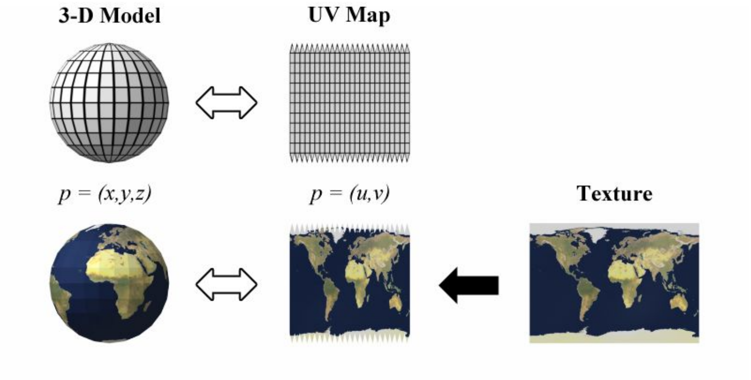
Texture Mapping

$$P = w_1 A + w_2 B + w_3 C$$

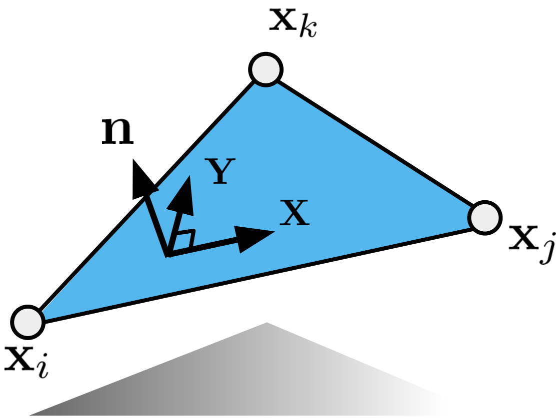
$$w_1 + w_2 + w_3 = 1$$







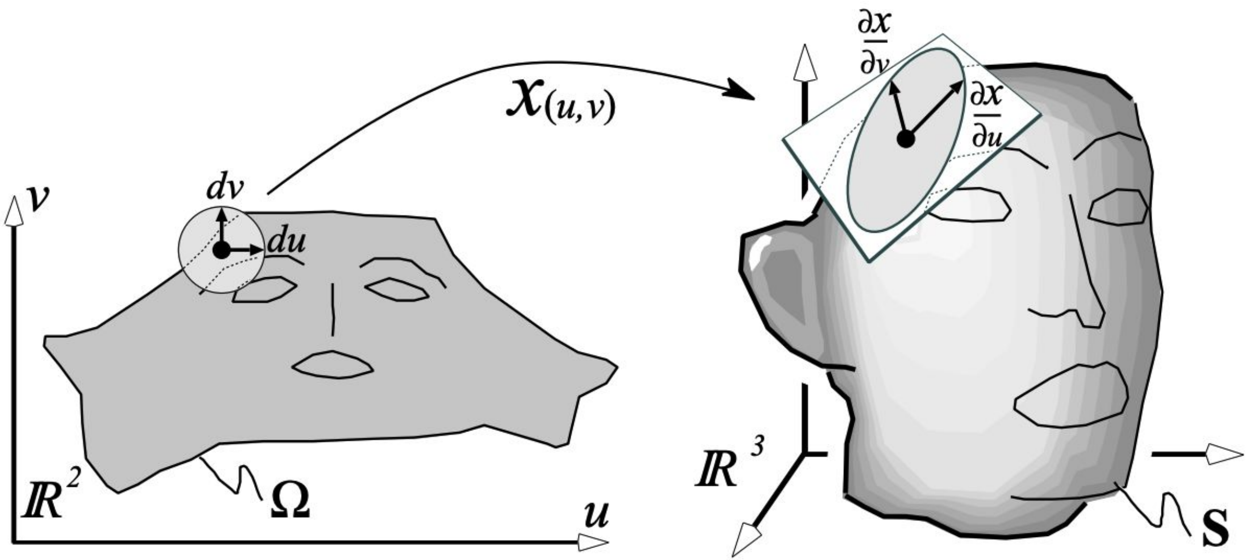
Gradient in a Triangle



$$\mathbf{X} = \frac{\mathbf{x}_j - \mathbf{x}_i}{||\mathbf{x}_j - \mathbf{x}_i||}$$

$$\mathbf{n} = \frac{\mathbf{X} \times (\mathbf{x}_k - \mathbf{x}_i)}{||\mathbf{X} \times (\mathbf{x}_k - \mathbf{x}_i)||}$$

$$\mathbf{Y} = \mathbf{n} \times \mathbf{X}$$



x(u,v)
v
dv
du
ℝ²
Ω
u
∂x/∂v
∂x/∂u
ℝ³
S

Constraints

Bijective (One-to-one mapping)

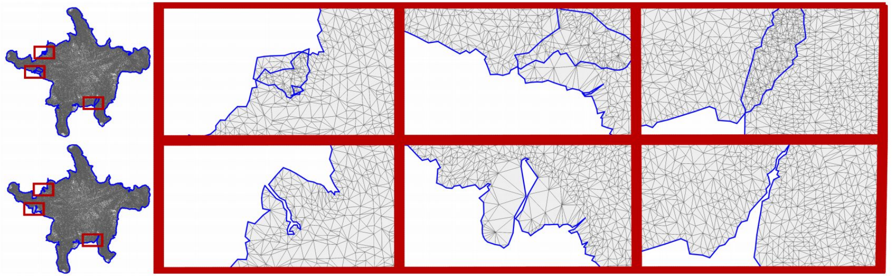
- The image of the surface in parameter space does not self-intersect
- The intersection of any two triangles in parameter space is either a common edge, a common vertex, or empty

Flip-free (inversion-free, foldover-free)

- The orientation of each triangle is positive

Locally injective (locally bijective)

- The orientation of each triangle is positive
- For boundary vertex, the mapping is locally bijective



[Smith et al 2015]